The value of $V_{ABCDEFGH}$ obtained from Eq. (14) is positive provided that the ordering of the corners A, B, C, D, E, F, G, H is as shown in Fig. 2. The corner points of any such hexahedron can be named in the way indicated in Fig. 2. The corner points A and B are first named so that AB is any side. Points D and E are then chosen so that AD and AE are sides and are such that AE turns towards AD when the hexahedron experiences a right hand rotation about the line AB. Points C, F, and H complete the three faces through A and the remaining point is G.

Addendum

The authors' attention has been drawn by J. H. B. Smith to the fact that the surface defined in Eq. (1) is a member of a one-parameter family of doubly-ruled quadric surfaces containing the quadrilateral ABCD. The pair of plane surfaces intersecting in the diagonal AC and the pair of plane surfaces intersecting in the diagonal DB are degenerate members of this family. The equation of a general member of the family is

$$[2p\xi\eta+(1-p)](\mathbf{RP}-\mathbf{RA})=(1-p)\xi(1-\eta)\mathbf{AB}$$
$$+(1-p)\eta(1-\xi)\mathbf{AD}+(1+p)\xi\eta\mathbf{AC} \qquad (15)$$

where p is the parameter and ξ,η are surface curvilinear coordinates.

The pair of plane surfaces intersecting in the diagonal AC are obtained when $p=+1$; the pair of plane surfaces intersecting in the diagonal BD are obtained when $p=-1$. The surface that is the same regardless of which corner of the quadrilateral is named A is obtained when $p=0$ and this is the case we have used in our Eq. (1) and is the case that we want in order that the hexahedral cells can be fitted together without any consideration of matching of diagonals of the face quadrilaterals.

Conclusions

An expression for the volume of a general hexahedron has been obtained. The faces of the hexahedron had first to be defined as doubly-ruled surfaces, but once that was done the actual volume, for a given set of eight corner points, is unique. Hexahedral cells so defined have the practical merit that they can be fitted together to fill the whole space without any consideration of matching of diagonals of the face quadrilaterals. Moreover the cell faces are completely smooth. The expression obtained is longer than that of Kordulla and Vinokur, but it need be evaluated only once for each cell in a finite volume method and the result stored. The penalty for evaluating the longer expression is tolerable when it is set against the advantage of the ease of fitting the cells together.

Acknowledgment

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The Position of Laminar Separation Lines on Smooth Inclined Bodies

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Nomenclature

i,j,λ	= integers
J	= number of vortex filaments starting from separation line
$K_\lambda(\xi)$	= coefficient in the equation of the vortex sheet [Eq. (1)]
$\bar{K}_\lambda(\xi)$	$=K_\lambda(2\lvert K_4\rvert)^{(2-\lambda)/2}$
$k_{\lambda,i}$	= coefficients in the expansions of $\bar{K}_\lambda(\xi)$, $\lambda=3,5$ [Eq. (2)]
L	= number of terms in the expansions of $\bar{K}_\lambda(\xi)$, $\lambda=3,5,-1$ [Eq. (2)]
ℓ_1	= x component of $\underline{\sigma}_1$
ℓ_R	= longitudinal reference length
$\underline{n}_0,\underline{n}_1$	= unit upstream normal vectors to parabolic cylinder and vortex sheet, respectively (Fig. 2)
$\underline{n}_2$	= unit vector, normal to vortex line and tangential to vortex sheet at P′, pointing downstream [Eq. (7) and Fig. 2]
δn_2	= infinitesimal length measured from P′ in the direction of $\underline{n}_2$
P	= typical point on separation line, on left-hand side facing upstream, with $x=\xi$ (Figs. 1 and 2)
P′	= typical point on sheet vortex line originating at P (Fig. 2)
Q	= general field point near vortex sheet (Fig. 1)
$\underline{r}_P,\underline{r}_Q$	= position vectors of P and Q, respectively, measured from origin of x, y, and z
x,y,z	= body Cartesian coordinates (Fig. 1)
y',z'	= local coordinates in plane normal to separation line at P, with origin at P (Fig. 1)
$\bar{y}',\bar{z}'$	= nondimensional coordinates $2\lvert K_4\rvert y'$ and $2\lvert K_4\rvert z'$, respectively
$\xi,\eta(\xi),\zeta(\xi)$	= x,y,z coordinates of P
$\underline{\sigma}_1,\underline{\sigma}_2,\underline{\sigma}_3$	= unit vectors parallel to separation line downstream tangent at P, Py', and Pz', respectively (Fig. 1)
$\underline{\tau}$	= unit downstream tangent to vortex line at P′ (Fig. 2)
$\psi(\xi)$	= vorticity flux function on vortex line originating at P
ω	= ± magnitude of surface vorticity vector on sheet at P′

Introduction

MANY calculations of separated flow from smooth elongated bodies have been made using inviscid potential-flow models (see for example, Refs. 1-4). Except for Ref. 4, all of these methods require prior specification of the separation lines.

Calculations of the laminar separation line itself have been made using boundary-layer calculations driven by unseparated

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flow pressure distributions[5,6] and, more recently, as part of full (but time-consuming), viscous treatments.[7]

Fiddes,[4] following a suggestion of Smith,[8] has used an inviscid slender-body method, with a viscous "triple-deck" treatment of the separation region, due to Smith,[9] to construct a method that finds the laminar separation lines as part of the solution. This seems to be the most satisfactory method to date.

Recently, the nonlinear vortex-lattice method for wings has been adapted to give vortex-filament iterative techniques (VFIT) for the separated flow past bodies,[7,11] when the separation line is known. It is proposed here to combine Fiddes' method[4] with a VFIT[7,11] and some ideas from Refs. 8-10 to build a method not restricted to slender conical shapes, as in Ref. 4. It may be possible to use a parabolic Navier-Stokes computation[12] in a restricted region near separation in place of the triple deck,[9] in which case the laminar-flow restriction could be removed as well.

In the VFIT, the free filaments originate on the separation line, so that, near the line, where the proposed method needs a good approximation to the flowfield, a poor one is obtained. To overcome this problem the author recalls Smith's[8] description of the genesis of the vortex-filament model at high Reynolds numbers. First, the thin separating vortex layers are idealized as vortex sheets, which are envelopes of sheet vortex lines (i.e., lines to which the sheet surface vorticity vector[13] is tangential), and then the constant vorticity fluxes along narrow ribbons of the sheets, which are bounded by vortex lines, are concentrated along their centerlines to form filaments, each of constant strength. This suggests the following hypothesis: *If a VFIT is used with the correct separation conditions[8] and sufficient filaments and segments in each filament are used, the final iterated filament shapes will lie close to vortex lines of the sheet that the filaments are replacing, even close to the separation line.*

This hypothesis enables us to use the segmented filament geometries and their strengths, as generated by the VFIT, to reconstruct the vortex-sheet shape near the separation line and the surface vorticity distribution on it, so that an accurate flowfield may be calculated, using the Biot-Savart law.[13]

Reconstruction of the Vortex Sheet near the Separation Line

In the following, separation on the body left-hand side is considered when facing upstream (see Figs. 1 and 2).

Information about the form of the vortex sheet in "open" separations near the separation line, as derived in Refs. 9 and 10, leads us to assume the following form for the sheet there:

$$z' = \sum_{\lambda=3}^{\infty} K_\lambda(\xi) y'^{\lambda/2} \tag{1}$$

where y' and z' are local planar rectangular coordinates in a plane intersecting the separation line orthogonally at $P(\xi,\eta,\zeta)$. For "smooth" separation, $K_3=0$, whereas K_4 always equals one-half of the curvature of the body section in the y'-z' plane at P. We form nondimensional quantities $\bar{y}'$, $\bar{z}'$, and $\bar{K}_\lambda$ with respect to the corresponding absolute values of the radius of curvature (if the body is locally flat at P, this procedure must be modified), truncate Eq. (1) to three terms, and assume that

$$\bar{K}_\lambda(\xi) = \sum_{i=0}^{L} k_{\lambda,i}(\xi/\ell_R)^i, \qquad \lambda=3,5 \tag{2}$$

L has been taken as 5 in all calculations carried out so far. Equation (1) now becomes

$$\bar{z}' - \tfrac{1}{2}\bar{y}'^2 \operatorname{sgn} K_4 = \bar{y}'^{3/2} \sum_{i=0}^{L} (k_{3,i} + \bar{y}' k_{5,i})(\xi/\ell_R)^i \tag{3}$$

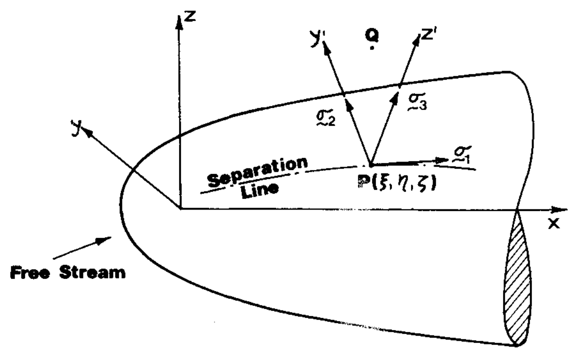

Fig. 1 Body and local coordinate systems.

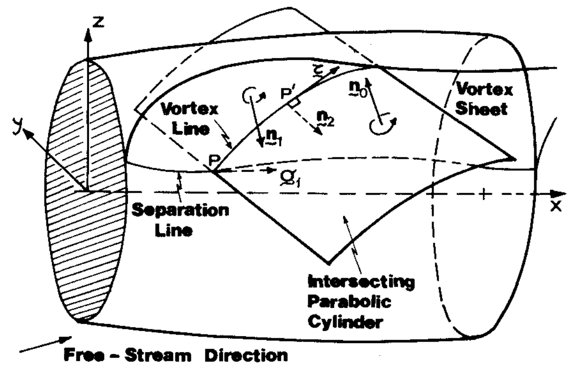

Fig. 2 Approximation to a vortex line.

To find y' and z' for some point Q near the separtion line, with body coordinates x, y, and z (Fig. 1), we evidently have the transformation

$$y' = (\underset{\sim}{r}_Q - \underset{\sim}{r}_P)\cdot\underset{\sim}{\sigma}_2, \qquad z' = (\underset{\sim}{r}_Q - \underset{\sim}{r}_P)\cdot\underset{\sim}{\sigma}_3 \tag{4}$$

with the point P and, hence, the y'-z' plane, identified by

$$(\underset{\sim}{r}_Q - \underset{\sim}{r}_P)\cdot\underset{\sim}{\sigma}_1 = 0 \tag{5}$$

since $\underset{\sim}{r}_Q - \underset{\sim}{r}_P$ is orthogonal to $\underset{\sim}{\sigma}_1$. Equation (5) has a unique solution for Q close to a gradually curved separation line, such as is found far from a blunt nose on an aerodynamic body. K_4, $\underset{\sim}{\sigma}_1$, $\underset{\sim}{\sigma}_2$, and $\underset{\sim}{\sigma}_3$ may be evaluated from the body and separation-line geometries.

If the VFIT utilizes J filaments from the separation line, the first two rows of nodal points *off* the line are used to yield, from Eq. (3), $2J$ equations for the $2(L+1)$ unknowns $k_{3,i}$ and $k_{5,i}$. Usually $J > L+1$, therefore, a least-squares routine is used to find the coefficients and, hence, the "near" sheet shape.

Reconstruction of the Sheet Surface-Vorticity-Vector Field

Once again, it is assumed that on each of the J filaments the point of origin on the separation line and two adjacent node points are given. This suggests seeking an approximation to the family of vortex lines on the sheet as the intersections between a family of infinite parabolic cylinders, with generators parallel to Oy and the sheet (see Fig. 2). It follows, from our basic hypothesis (see Introduction), that typical members of the family of parabolas may be found from each of the aforementioned sets of three points, allowing the complete family to be found by interpolation, thus defining the family of vortex lines.

Referring to Fig. 2, the unit tangent to the vortex line is

$$\underset{\sim}{\tau} = (\underset{\sim}{n}_0 \times \underset{\sim}{n}_1)/|\underset{\sim}{n}_0 \times \underset{\sim}{n}_1| \tag{6}$$

where $\underset{\sim}{n}_0$ and $\underset{\sim}{n}_1$ are known, respectively, from the geometry of the parabolic cylinder and vortex sheet at P′, the point in question. The unit downstream normal to the vortex line, which is also tangential to the sheet, is

$$\underset{\sim}{n}_2 = \underset{\sim}{\tau} \times \underset{\sim}{n}_1 \tag{7}$$

and the sheet vorticity vector at P′ is given by $\omega\underset{\sim}{\tau}$, where

$$\omega = \frac{d\psi}{d\xi}\,\frac{d\zeta}{dn_2} \tag{8}$$

and $\psi(\xi)$ is a scalar vorticity-flux function representing the vorticity flowing onto the sheet across the separation line, up to $P(\xi,\eta,\zeta)$, i.e.,

$$\psi(\xi) = \int_{\xi_0}^{\xi} [\omega(\underset{\sim}{n}_2 \cdot \underset{\sim}{g}_1)/\ell_1]\,dx \tag{9}$$

A table of values for ψ may be constructed from the J filament strengths calculated by the VFIT, so that $d\psi/d\xi$ can be found numerically. The ξ corresponding to the vortex line through P′ is found by identifying the particular parabolic cylinder on which it lies.

Corrected Velocity Field on the Body near the Separation Line

By replacing the field of the two rows of filament segments nearest to the body by the field of the reconstructed "near" sheet, found using the Biot-Savart law,[13] the required corrected field is obtained, since "distant" parts of the sheet are still adequately represented by the filaments.

Conclusion

A method of calculating laminar separation lines on general smooth aerodynamic bodies is proposed, based on adaptation of an existing slender-body method.[4] The fundamental assumption of the method remains to be checked through suitable test cases, but certain aspects of feasibility and accuracy have been checked and found to be satisfactory.

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Transonic Small-Disturbance Theory for Dusty Gases

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I. Introduction

THE flow of gases containing small particles (dust) is interesting to study and is important in many practical situations. A method for analyzing the effect of the dust on flows in the transonic regime is presented in this Note.

Marble[1] derived a model in which the gas and dust exchange heat and momentum. The equations of conservation of mass, momentum, and energy for each material are simplified by assuming low volumetric concentations of a relatively heavy dust. This set of dusty gas equations is then further approximated by assuming strong coupling between the materials. The resulting system of equations is analogous to the equations for the adiabatic motion of an ideal gas, except that the ratio of specific heats γ and density ρ are modified to reflect the heat capacity and density of the dust. Marble also showed that this generalized gas supports discontinuities in properties (generalized shocks) which consist of a shock in the gas, followed by a relaxation back to equilibrium (both thermal and mechanical) of the dust particles.

Marble's model is applied to transonic small-disturbance theory. It is shown that the effect of the dust is to modify the flow in such a way that it is equivalent to the flow of a clear gas at a different freestream speed around an airfoil of a different thickness.

This work is complementary to that of Barron and Wiley,[2] who derived a version of small-disturbance theory for hypersonic dusty gases. Applying the Newtonian approximation $\gamma \to 1$ and $M \to \infty$, they are able to work out an interesting solution for wedge-shaped bodies. The present work does not use this approximation; instead, a similarity transformation is employed which relates the equations of dusty gases corresponding to those for dust-free flow.

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